An Introduction to Mathematical Proofs

Uncountable

Who? Fahad Hossaini

When? Whenever You Watch

$\mathbb{Q}$

We listed every element in $\mathbb{Q}$ last video.

Before we proceed with today's topic, I'll introduce you to a cool technique to prove cardinality.

Abusing Bijections

Imagine the following way to list the rationals.

$$\ldots, \underbrace{-\frac{3}{1}, -\frac{1}{3}}_{-G_4}, \underbrace{-\frac{1}{2}, -\frac{2}{1}}_{-G_3}, \underbrace{-\frac{1}{1}}_{-G_2}, \underbrace{\frac{0}{1}}_{G_1}, \underbrace{\frac{1}{1}}_{G_2}, \underbrace{\frac{1}{2}, \frac{2}{1}}_{G_3}, \underbrace{\frac{3}{1}, \frac{1}{3}}_{G_4}, \frac{1}{4}, \ldots$$

The issue is we can't map this sequence to the naturals, because it goes to infinity both to the left and right.

However, we can map this sequence to the integers. Let $f : \mathbb{Z} \to \mathbb{Q}$ and $f(0) = \frac{0}{1}$. Then, map in both directions. So, $f(-1) = \frac{-1}{1}$, $f(-2) = \frac{-2}{1}$, $f(1) = \frac{1}{1}$, $f(2) = \frac{2}{1}$ and so on.

This is a bijection. So, $|\mathbb{Q}| = |\mathbb{Z}|$. But we know that $|\mathbb{Z}| = |\mathbb{N}|$. Thus, $|\mathbb{Q}| = |\mathbb{N}|$.

Key Takeaway

We already showed that if $f : A \to B$ and $g : B \to C$ are both bijective and $h : A \to C$ by $h = g \circ f$, then h must be bijective.

But now, we also notice that $|A| = |B|$ by f, $|B| = |C|$ by g and $|A| = |C|$ by h.

Thus, if $f : A \to B$ is bijective, to show that some other set C has property: $|C| = |A|$, it's sufficient to show that $|C| = |B|$.

Okay, Now Practical

Our end goal is find $|\mathbb{R}|$.

We'll show that $|\mathbb{R}| = |(0,1)|$, the open interval.

Consider the function $f(x) = arctan(x)$. f is bijective when $f : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$. This depends on $tan(x)$ being bijective on $f^{-1} : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$.

You can rigourously prove that the above function(s) are bijective. We'll take it as granted. You can do it in the worksheets [insert rofl emoji].

Now, we'll change f. If $f(x) = \frac{2}{\pi} arctan(x)$, then f is bijective when $f : \mathbb{R} \to (-1, 1)$. You can also prove this.

This is a very interesting property...

Scaling Functions

We'll prove the claim from the previous slide.

If $f : \mathbb{R} \to (a, b)$ is bijective, then $c \cdot f : \mathbb{R} \to (ca, cb)$ is also bijective, where $c \cdot f$ multiplies f by c. For example, if $f(x) = y$, then $c \cdot f(x) = c\dot{y}$.

You should be able to prove this. Try to figure out the steps. We'll prove it in the next two slides.

Scaling Functions Proof

If $f : \mathbb{R} \to (a, b)$ is bijective, then $c \cdot f : \mathbb{R} \to (ca, cb)$ is also bijective, where $c \cdot f$ multiplies f by c. For example, if $f(x) = y$, then $c \cdot f(x) = c\dot{y}$.

Proof. Regarding injectivity, let $x_1 \neq x_2 \in \mathbb{R}$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then,

$$\begin{aligned} x_1 &\neq x_2 \\ f(x_1) &\neq f(x_2) && \text{Since f is bijective} \\ y_1 &\neq y_2 \\ c \cdot y_1 &\neq c \cdot y_2 \\ c \cdot f(x_1) &\neq c \cdot f(x_2) \end{aligned}$$

□

Scaling Functions Proof

If $f : \mathbb{R} \to (a, b)$ is bijective, then $c \cdot f : \mathbb{R} \to (ca, cb)$ is also bijective, where $c \cdot f$ multiplies f by c. For example, if $f(x) = y$, then $c \cdot f(x) = c\dot{y}$.

Proof. Continued.
Regarding surjectivity, let $y \in (ca, cb)$. We also know that $\frac{y}{c} \in (a, b)$. Then, let $x = \frac{y}{c}$. We have:

$$c \cdot f(x) = c \cdot \frac{y}{c} = y.$$

Thus, $c \cdot f(x)$ is bijective. $\square$

Translating Functions

So now, $f(x) = \frac{2}{\pi} arctan(x)$, which is bijective when $f : \mathbb{R} \to (-1, 1)$. For our future sanity, let $f(x) = \frac{4}{\pi} arctan(x)$, which is bijective when $f : \mathbb{R} \to (-\frac{1}{2}, \frac{1}{2})$

Now, we'll change f again. Let $f(x) = \frac{4}{\pi} arctan(x) + \frac{1}{2}$. This is bijective when $f : \mathbb{R} \to (0, 1)$.

Once again, I'll leave this for you to prove. It's very similar to the proof for scaling functions.

Conclusions

1. When bijective functions go from the reals to an interval, we can scale and translate the function and as long as we accordingly change the interval, the function stays bijective.

2. Since $f(x) = \frac{4}{\pi} arctan(x) + \frac{1}{2}$ is bijective when $f : \mathbb{R} \to (0, 1)$, we have that: $|\mathbb{R}| = |(0, 1)|$.

So, all we need to do is come up with a way to count every number between 0 and 1, and we'll have that $(0, 1)$ is countable.

Why The Reals Are Unique

The issue is this: Every real number between 0 and 1 can be written to have countably many decimals. For example, $\frac{1}{5} = 0.200000000\ldots$

We could have patterns that don't repeat, for example, $0.68290847126623701786567842476795 6\ldots \in (0,1)$ and we don't know what the rest beyond the dots will look like.

But, you know what, let's try!

The Idea

We want to show that $|\mathbb{N}| = |(0,1)|$.

Assume that we can list every number in $(0,1)$. We'll have to figure out how to do this formally later, but let's continue!

Then, let y_1 be our first number. For example, it could be $0.325701756104562\ldots$ And let y_2 be our second number. For example, it could be $0.191111111\ldots$

Then, define $f : \mathbb{N} \to (0,1)$ by $f(n) = y_n$. This should be injective as long as we don't allow duplicates.

But regarding surjectivity...

A Problem...

We're going to construct an $x \in (0, 1)$.

Let x_1 be the first digit of y_1 after the decimal plus 1. For example, if y_1's first digit is 3, then $x_1 = 4$. In the case that a digit is 9, we go back to 0. Let x_2 be the second digit of y_2 after the decimal plus 1. So, $x_2 = 0$. Do this for every natural number. Thus, x_n is the nth digit of y_n after the decimal plus 1.

Then, let $x = 0.x_1x_2x_3\ldots$. This is *not* multiplication! This is concatenation! So, the first digit after the decimal of x is x_1, the second digit after the decimal is x_2 and so on.

Do you see why x is unique?

A Visual

$$
\begin{aligned}
y_1 &= 0.325701756104562\ldots \\
y_2 &= 0.191111111111111\ldots \\
y_3 &= 0.708950345327463\ldots \\
y_4 &= 0.091095468354291\ldots \\
y_5 &= 0.874157567896514\ldots \\
y_6 &= 0.678981384712222\ldots \\
y_7 &= 0.463724987532496\ldots \\
y_8 &= 0.675268456894654\ldots \\
y_9 &= 0.197846352794412\ldots \\
\vdots &= \vdots
\end{aligned}
$$

$$x = 0.\ 4\ 0\ 9\ 1\ 6\ 2\ 0\ 6\ 3\ \cdots$$

What's Unique About x?

Let's compare x to the rest of our list.

Clearly, $x \neq y_1$ as $x_1 \neq$ the first digit after the decimal of y_1. Remember, if any digit is different, then the number can never be equal.

But we also have that $x \neq y_2$ because of x_2. Same with y_3 and so on.

So, $x \neq y_n$ for all $n \in \mathbb{N}$.

Wait a second... we just made a new number in $(0, 1)$ that wasn't in our original list...

Uhhhhh......................

Hold.

Didn't we assume every number in $(0, 1)$ is in our list? And then we made a new number that wasn't listed? So which natural number maps to x?

None of them...

Wait, didn't we just prove that f can never be surjective?

Wow...

We just showed that $|\mathbb{N}| < |(0,1)| = |\mathbb{R}|$. What we did was a proof by contradiction. Essentially,

Proof. Assume for the sake of contradiction that $f : \mathbb{N} \to (0,1)$ is bijective. Then, we can produce $x \in (0,1)$ so that $f(n) \neq x$ for every $n \in \mathbb{N}$ (remember, f maps to y_n but $x \neq y_n$ for every natural n).

Thus, f can never be surjective, so $|\mathbb{N}| < |(0,1)|$. $\square$

We can *never* enumerate or list every real number, let alone every number between 0 and 1.

The Gravity Of The Situation

We just showed that $\mathbb{N}$ and $\mathbb{R}$ have different cardinality. But both are infinite?

We proved that there are different sizes of infinity.

Just think about that for a second. We can find a larger infinity. An unlistable, unenumerable, uniterable, uncountable infintity.

$|\mathbb{R}|$

We say that $\mathbb{R}$ has uncountable cardinality.

Cantor's Diagonal Argument

The technique we used is called Cantor's Diagonal Argument.

No one believed him when he published his proof claiming that there were different sizes of infinity. But, it's commonly accepted as fact nowadays.

Infinite PHP Part 2

If we have countably many pigeonholes but uncountably many pigeons, what can we deduce?

We can deduce the following: At least one pigeonhole will have uncountably many objects.

intuition: Countable unions of countable sets is countable, say $S_1 \cup S_2 \cup S_3 \cup \cdots$ where $|S_n|$ is countable for all $n \in \mathbb{N}$.

We can arrange the sets in a grid pattern and dovetail. First row is S_1, second row is S_2, third row S_3 and so in. Remove duplicates and bam, countable.

Thus, some S_i must have uncountable cardinality.

That's It For This Unit

One Last Note: I want you understand the severity of this.

As humans, we fail to comprehend large things. A number doesn't do justice to the size of the universe. Likewise, it doesn't do justice to the speed of light.

If we fail to comprehend large, what chance do we have to understand countable infinity? What about uncountable?

I acknowledge the fact that I'll never be able to wrap my head around this fact. But despite that, it's beautiful.

If nothing I've shown or taught sticks with you, remember just this one: Math is beautiful, even though we can't comprehend it in its entirety.