An Introduction to Mathematical Proofs

Summations

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When? Whenever You Watch

Phew...

After all that logic, we need a little break. The next three episodes are connected, and then after that, we can return to the land of functions to start our bridge into the second half.

Closed Forms

How would you calculate $1 + 2 + \cdots + 100$?

If I gave you some money, you might plug it into a calculator. But what if I ban calculators? Then you'll write it out by hand, slaving away.

But is there a more efficient way to solve this. What about the sum of 1 to 1000?

An Idea

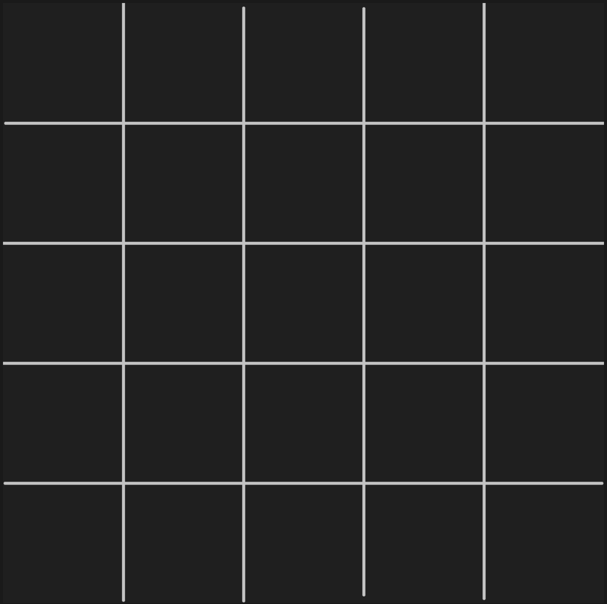
Let's return to our 'sum to 100' question. What we could do instead is match up the last number and first number, and keep proceeding inwards.

For example, 100 and 1, 99 and 2, 98 and 3, and so on. Then, we'll notice that these add up to 101, and there are 50 copies of these. So, the final sum is 5050.

And yes, this works for odd numbers as well. If we sum up to 99 instead, each pair adds up to 100 and there are 49 pairs. But the middle one is 50, so it acts like half of a pair. So $100 \cdot 49.5 = 4950$

So, we say that $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.

A Picture



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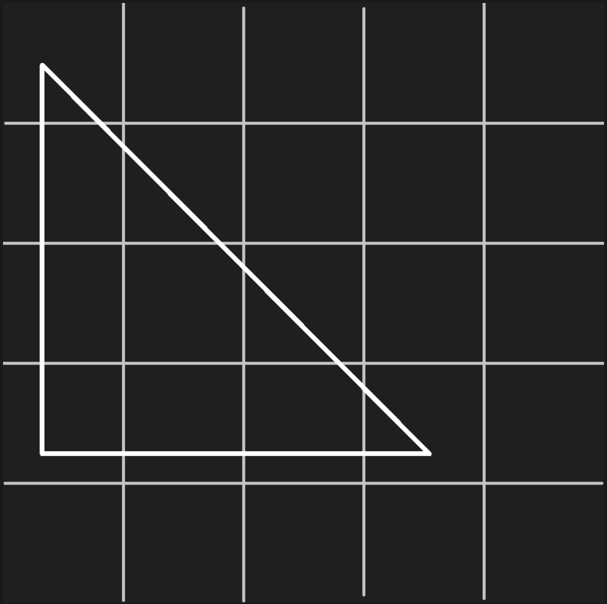

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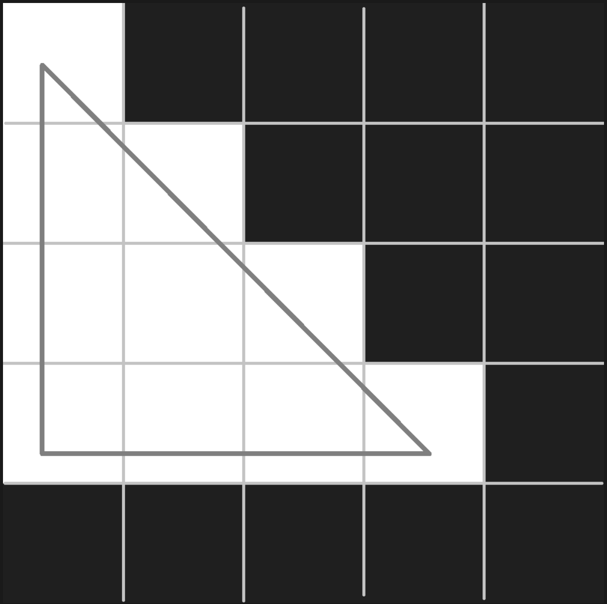

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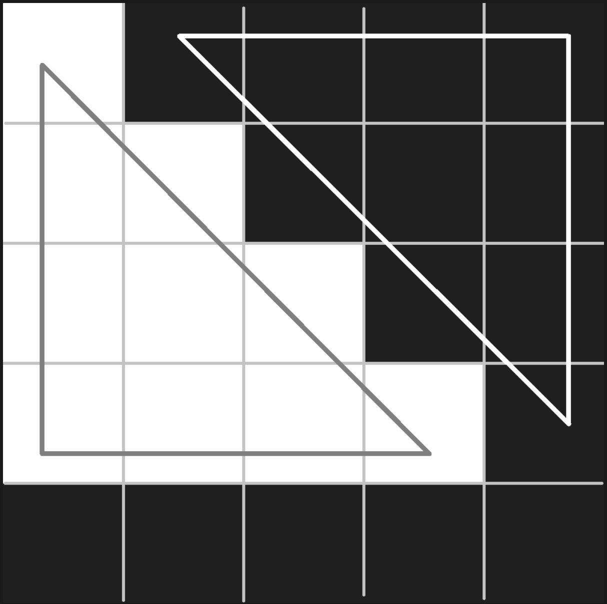

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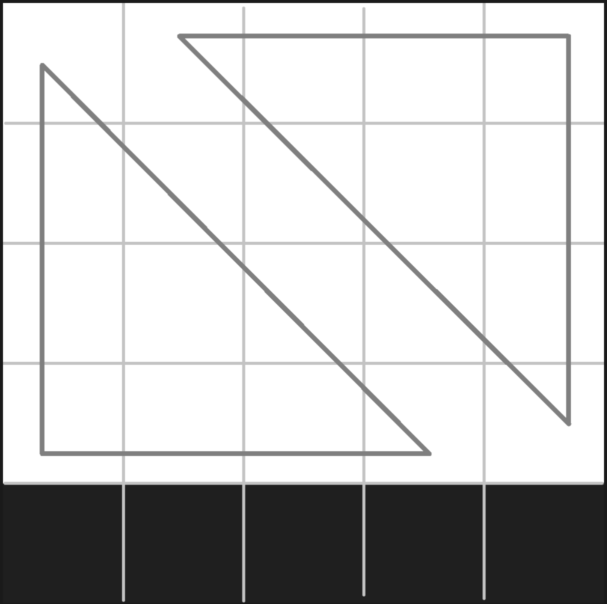

Another Way To Conclude:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

Sigma Notation

We can rewrite $1 + 2 + \cdots + k$ as:

$$\sum_{i=1}^{k} i$$

The number underneath the capital sigma is called our index. We let our index start at whatever number we want. In this case, 1.

Every time we repeat addition, we increase i by 1.

The number above the sigma is where we stop (end included). Or, the last number our index goes to. In this case, $i = k$ is the last time we add.

To the right of the sigma is the expression we keep repeatedly adding.

Why This Notation?

It's compact. While I like writing $1 + 2 + \cdots + k$, it gets long to write. And they're some cool properties of this notation. But before that, practice!

Example What is $\sum_{i=3}^{6} 2i^2$?

Example What is $\sum_{i=2}^{10} i^2 + i$?

Answers!

Example $\displaystyle\sum_{i=3}^{6} 2i^2 = 2\cdot 3^2 + 2\cdot 4^2 + 2\cdot 5^2 + 2\cdot 6^2 = 2(3^2+4^2+5^2+6^2)$

Example $\displaystyle\sum_{i=2}^{10} i^2 + i = (2^2+2) + (3^2+3) + \cdots + (10^2+10) =$

$$2^2 + 3^2 + \cdots + 10^2 + 2 + 3 + \cdots + 10 = \sum_{i=2}^{10} i^2 + \sum_{i=2}^{10} i?$$

Some Properties

Factoring: $\forall c \in \mathbb{R}, \sum_{i=j}^{k} c \cdot i = c \cdot \sum_{i=j}^{k} i$

Split expressions: $\forall a, b$, where a, b are expressions,
$\sum_{i=j}^{k} a + b = \sum_{i=j}^{k} a + \sum_{i=j}^{k} b$

Split sums: $\forall k_1 \in \mathbb{N}$ where
$j < k_1 < k, \sum_{i=j}^{k} i = \sum_{i=j}^{k_1} i + \sum_{i=k_1+1}^{k} i$

Constants: $\forall x \in \mathbb{R}, \sum_{i=j}^{k} x = (k - j + 1)x$

Prove Them!

Prove these properties! They're surprisingly easy to do. Expand out the expression and rearrange. That's all there is!

Now, Back To Closed Forms

Now that we have the Sigma Notation, how can we prove our closed forms? Or, formulas?

Well, it's a little uninspired. In fact, we lose part of the intuition while proving these closed forms.

But it's a price we have to pay to motivate and understand our tools. Once again, we are developing tools, and I'm providing you one example of why we would want the tool we'll talk about in the next video.