An Introduction to Mathematical Proofs

Inequalities

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When? Whenever You Watch

Some Basic Rules

- 1: $x^2 \geq 0$
- 2: If $a > b$ and $b > c$, then $a > c$
- 3a: If $a > b$ and $c > 0$, then $c \cdot a > c \cdot b$.
- 3b: If $a > b$ and $c < 0$, then $c \cdot a < c \cdot b$
- 3c: If $a > b$, then $a + c > b + c$
- 4: If $a > 0$, then we can find a unique non-negative number called $\sqrt{a}$ so that $\sqrt{a}^2 = a$.

Similar Versions to the Rules Above

For 2,3 we can switch inequality directions: If $a<b$ and $b<c$, then $a<c$

For 2,3 we can also use non-strict inequalities: If $a \geq b$ and $b \geq c$, then $a \geq c$

Cancellation

Example Show that $a + b > c$ is the same as $a > c - b$.

Proof.

We have:

$$\begin{aligned} a + b &> c \\ a + b - b &> c - b && \text{By rule 3c} \\ a &> c - b && \text{By ...?} \end{aligned}$$

□

Note: This box symbol means we completed the proof. We want to tell our readers when a proof starts and ends, hence box.

Basic Number Facts

Example If $0 < a < b$, then show that $a^2 < b^2$.

Proof.

Multiply by b on both sides: $a \cdot b < b \cdot b = b^2$

But since $a < b$, we have that: $a^2 = a \cdot a < a \cdot b$

Combining these, we get: $a^2 < a \cdot b < b^2$

By rule 2, we have that: $a^2 < b^2$

□

Exercise

If $0 < a < b$, then show that $\sqrt{a} < \sqrt{b}$.

How Do We Turn Our Intuition Into Algebra?

We'll simplify to the one-dimensional case and proceed from there.

A Quick Note On Trust

I want to make sure you undertand what's happening in the course.
So, if something doesn't make sense or you need more time to digest something, verify it or take your time! No rush.

Do not blindly trust me, convince yourself that what I am saying makes sense, and if it does not, write it down! You can ask or later find our where the misunderstanding is.

The Triangle Inequality

$$|x + y| < |x| + |y|$$

Absolute Values

Definition

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- 1: $a \leq |a|$
- 2: $|a|^2 = a^2$
- 3: $\sqrt{a^2} = |a|$
- 4: $|ab| = |a| \cdot |b|$

Our First 'Tough' Proof

We want to show the following: For all values of x and y, we have that: $|x+y| \leq |x|+|y|$.

This means, we want to start from something true and reach this conclusion.

But where do we start?

Our First 'Tough' Proof

If we go backwards using logical steps, then we should be able to construct our proof backwards.

Let's play around with some ideas.

Playing Around

We have the following: $|x + y| \leq |x| + |y|$. Let's square both sides and see where it goes:

$$
\begin{aligned}
|x + y| &\leq |x| + |y| \\
|x + y|^2 &\leq (|x| + |y|)^2 \\
|x + y|^2 &\leq |x|^2 + 2|x||y| + |y|^2 \\
(x + y)^2 &\leq x^2 + 2|x||y| + y^2 \\
x^2 + 2xy + y^2 &\leq x^2 + 2|x||y| + y^2 \\
2xy &\leq 2|x||y| \\
xy &\leq |xy|
\end{aligned}
$$

Formal Proof

Here's the complete proof:

Proof.

We know that $xy \leq |xy|$ by rule 1 of absolute values. Then, we multiply by two and expand the right hand side by rule 4 to get:

$$2xy \leq 2|xy|$$

Now, we add x^2 and y^2 to both sides. Then, we note that since $x^2 = |x|^2$ and $y^2 = |y|^2$ by rule 2, we have the following:

$$x^2 + 2xy + y^2 \leq |x|^2 + 2|x||y| + |y|^2$$

Formal Proof

Proof. Continued

$$x^2 + 2xy + y^2 \leq |x|^2 + 2|x||y| + |y|^2$$

Now, we use the algebraic identity:
$(x + y)^2 = x^2 + 2xy + y^2$ on both sides to get:

$$(x + y)^2 \leq (|x| + |y|)^2$$

Finally, after square rooting both sides, we can apply rule 3 to the left side to get:

$$|x + y| \leq |x| + |y|$$