An Introduction to Mathematical Proofs

Induction Variants

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When? Whenever You Watch

Recap: Induction

Last video, we learned about Induction with our best friend, Benny the frog.

Recall our motivation: Induction is a tool to solve problems. While we motivated induction with solving closed forms, we can use induction in many ways. Here's an example.

Prime Factorization

Show that every natural number greater than 2 is either a prime or can be factored into primes.

Before we prove this, let's run some examples.

Examples

2 is prime. Good!

3 is prime. Good!

$4 = 2 \cdot 2$, so it's composed of prime factors.

5 prime, $6 = 3 \cdot 2$, so good.
Okay, it seems like the claim is true but induction might not work to prove this, but it does! Let's analyze our problems and how to fix them.

Problem 1: Base Case

We need to show our claim for every natural number greater than 2, but induction only works on base case 1?? How to solve??

Okay, clearly we just start at 2. But this gives us an important insight: we can let our base case be whatever we want!

This will be useful for later questions!

Problem 2: Induction Hypothesis

In induction, we only worry about jumping from one lilypad to the next one. Or, assume our claim holds for k and show it for $k + 1$.

But in this case, 4 relies on 2 being prime, so we need more than one step back...

But this actually isn't a problem! We'll diagram it.

Diagram Time!

We start with lilypad 2. We know that lilypad 2 and 3 are covered, so let's color those in.

Benny Can't Jump

But here we see the dilemma. Benny can't jump from lilypad 3 to lilypad 4...

But wait! Benny *can* jump from lilypad 2 to lilypad 4! And since he got to lilypad 2, since he definetely got to 3, there's no issues!

So 4 is good to go!

5's a Gimme

Let's color in 5 since it's prime. Then, we'll tackle 6.

6 was scared of 7

Now, Benny can't jump from either 4 or 5 to 6. But it doesn't matter, because he can jump from 2 and 3 to 6. Yes, he needs multiple lilypads to cross that gap.

All that matters is this: Even if Benny needs multiple lilypads from beyond the previous lilypad, it's still fine because we were able to get on all the previous lilypads!

Strong Induction

Enter Strong Induction. Strong Induction is Induction but with the following change:

The induction hypothesis doesn't just include lilypad k but every lilypad before and including k. So we can use multiple lilypads from anywhere before k.

And this doesn't affect the truth of our proof! All we're using is all the previous knowledge we've gained from all the lilypads, not just the last one!

Proving Prime Factorization

Now, let's prove prime factorization for all $n > 2 \in \mathbb{N}$ using Strong Induction! Namely, every natural greater than 2 is either prime or factors into primes.

Proof. We start by considering our base case, $k = 2$. Since 2 is prime, our base case holds.

Now we proceed with our inductive step. Assume that every number from 2 to k is either prime or can be factored into primes for some $k \in \mathbb{N}$. Then, we'll show that $k + 1$ is either prime or can be factored into primes.

Some people like the following notation: Let $P(k)$ be the claim that k is prime or can be factored into primes (in this case, our base case is $P(2)$). Then, our induction hypothesis can be: Assume that $P(2), P(3), \ldots, P(k)$ is true for some $k \in \mathbb{N}$. Show that $P(k + 1)$ is also true.

Inductive Step

Proof. Continued.
Remember, in our induction hypothesis, we are allowing all the lilypads before k to be part of our induction hypothesis.

Now consider $k+1$. If it's prime, the claim holds, or $P(k+1)$ is true. If $k+1$ is not prime, then we can divide it so that:
$k+1 = l \cdot m$ for some $1 < l, m < k+1$. Since both l and m are less than $k+1$, we have m and l are both either prime or can be factored into primes by our induction hypothesis. Thus, $k+1$ can be factored into primes.

Or in other words, since $P(m)$ and $P(l)$ is true, we must have that $P(ml) = P(k+1)$ is true.

Last But Not Least

Proof. Continued.

Since our inductive step is true, in combination with our base case, every natural number is either prime or can be factored into primes. Or, $P(n)$ is true for all $n \in \mathbb{N}$. □

Here, I provided two ways you can think of the inductive step. Either directly or using $P(k)$ for different values of k.

There's actually a name for $P(k)$. It's called the predicate. So for induction, you can either solve it directly, or show that the predicate holds for all natural numbers.

Remember, if you use the predicate path, you define the predicate depending on the problem. I personally prefer direct induction, so I'll stick with that.

Midpoint Break

In our first problem, we adapted induction by changing the base case and by strengthing our induction hypothesis leading to strong induction. Interestingly, both induction and strong induction have the same strength, but strong induction gives us more information.

But we can still adapt the tool of induction in another way, which we'll highlight in the next problem.

We Might Need Benny's Friend Here...

Show that every number greater than 2 can be written as a sum of 2s and 3s. For example,
$41 = (11 \cdot 3) + (4 \cdot 2)$.

As always, let's convince ourselves this is true by doing examples.

2 and 3 are fine. $4 = 2 + 2$. Good. $5 = 3 + 2$. Good.
$6 = 3 + 3 = 2 + 2 + 2 = 2 + 4$. Good.
$7 = 3 + 2 + 2 = 3 + 4 = 5 + 2$. Good.
$8 = 2 + 2 + 2 + 2 = 3 + 3 + 2 = 3 + 5 = 6 + 2$. Good.

Okay, that's a lot of information but do you see the pattern?

Insight

We can write every n as $(n-2)+2$. But wait, $(n-2)$ is only two lilypads back. So we'll still need to use strong induction, but there's an issue in our induction step.

Intuitively, we can only go from lilypad k to lilypad $k+2$! So how are we gonna reach every lilypad?

More Diagrams???

Let's put Benny on lilypad 2 and see where this goes.

This Reminds Me Of Leapfrog...

Benny jumps to lilypad 4.

This Reminds Me Of Leapfrog...

Benny jumps to lilypad 6.

This Reminds Me Of Leapfrog...

Benny disappears off screen (don't worry, he's doesn't die off screen unlike a certain somebody from JJK...).

But who's gonna deal with the odd numbers?

Tiffy... OMG, Benny!!!!!!!

Enter Tiffy. • *ribbit*

She's gonna deal with the odd numbers!

Tiffy The Goat For Real For Real

Let's put Tiffy on pad 3 and get her hopping.

Hop To The Beat!

Hop To The Beat!

Just like that, every lilypad is covered! Okay, let's recap what just happened.

The Big Idea

We noticed that our induction hypothesis doesn't prove the next case but the case after. Or in predicate terms, $P(k) \Rightarrow P(k+2)$.

But with only one base case, we only hit the even numbers.

So if we imagine our base case as frogs, one base case (Benny) and its inductive step covers all the even numbers, while the second base case (Tiffy) and its inductive step cover all odd numbers.

Yes, this is a matter of base cases. So here's the extra thing we can change about induction: Multiple base cases. Now, let's prove this!

Proof!

Prove that every number greater than 2 can be written as a sum of 2s and 3s.

Proof. We start with our base cases. Our first base case is 2 while our second base case 3. Clearly, both can be written as a sum of 2s and 3s. Proceed with the inductive step.

Assume that every number from 2 to k can be written as a sum of 2s and 3s. Or, let 'k can be written as a sum of 2s and 3s' be our predicate. Then, if $P(2), P(3), \ldots, P(k)$ is true for some $k > 3 \in \mathbb{N}$. Show that $k + 1$ can be written as a sum of 2s and 3s, or, $P(k + 1)$ is true.

Inductive Step

Proof. Continued.

We'll prove our inductive step directly. We have: $k+1=(k-1)+2$. By our induction hypothesis, $k-1$ can be written as a sum of 2s and 3s, so clearly, $k+1$ can also be written as a sum of 2s and 3s. Thus, our inductive step holds and the proof is complete. $\square$

Why A Second Base Case?

What's interesting is that it doesn't seem clear why we need a second base case. It seems claiming strong induction should be fine so what's the issue? Even intuitively, we saw that we need a second base case, or Tiffy, but where in the proof does that appear?

If we try to prove 3 using induction, we can't use this method since $3 - 2 = 1$! While it's super sneaky, the reason for our second base case is to ensure that our inductive step is performed on the odd numbers and by extension, all numbers greater than 3.

This Is Why Intuition Is Important!

Intuition allows us to understand the small details that some people might forget to 'remember' when doing proofs. So when we do our proofs, understanding is important because it ensures we don't forget crucial steps or follow a recipe blindly and can't tackle a slightly harder/different problem.

A Small Detail

In all our induction proofs, in our inductive step, we say assume the claim holds from some $k \in \mathbb{N}$. This is the same as saying Benny can reach lilypad k.

If we incorrectly assume that the claim holds for all $k \in \mathbb{N}$, that's wrong because that's the same as saying Benny can reach every lilypad.

Apparently, this is a trap many people fall into, but if you remember Benny, this shouldn't be an issue. Just be careful!

Recap!

That's all there is to induction. Remember Benny and the intricacies of lilypad jumping. Then, you'll be fine!

We learned closed forms and the classic induction. Then, we learned ways to adapt induction to solve our problems by:

Changing where our base case starts.

Using every/many lilypad(s) before k and learning about strong induction.

Having multiple base cases.