An Introduction to Mathematical Proofs

Dovetailing

Who? Fahad Hossaini

When? Whenever You Watch

Recap

Last video, we introduced cardinality as a concept. Then, we looked at the cardinality of the evens, extended PHP and ended by looking at the cardinality of the integers.

We'll be exploring two interesting sets and their cardinality. Namely, $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Q}$.

A Useful Conclusion

Prove that if $A \subseteq B$, then $|A| \leq |B|$.

Proof. Since $A \subseteq B$, we can find an injective function from A to B via the map $f(x) = x$. Clearly, every element in A is in B so this is well defined.

Since f exists, we could never have that $|A| > |B|$. Thus, we must have that $|A| \leq |B|$. $\square$

Set Operations

Before we investigate $\mathbb{N} \times \mathbb{N}$, let's look at our set operations and see what cardinality results make sense.

First, union. What can we say about $|A \cup B|$ when we know $|A|$ and $|B|$?

Union

First, union. What can we say about $|A \cup B|$ when we know $|A|$ and $|B|$?

By definition, $A \cup B = \{x : x \in A \vee x \in B\}$. Thus, $A \subseteq A \cup B \Rightarrow |A| \leq |A \cup B|$. Remember, even if we have subsets with no equality, we can still share equality.

Likewise, $|B| \leq |A \cup B|$ so $|B| \leq |A \cup B|$.

You can think of finite sets. But we'll do an infinite example. Let E be all natural even numbers and O be all natural odd numbers. Then, $|E| = |O| = |\mathbb{N}|$.

Intersections

Regarding intersections, our conclusions are reversed. Why? $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Thus, $|A \cap B| \leq |A|$ and $|A \cap B| \leq |B|$.

Think of some examples! Remember, since both conclusions must be true, if one set has countable cardinality while the other is finite, then the intersection must have finite cardinality.

Difference and Complement

Difference follows similarly to the case for Intersection.

Regarding Complement, it becomes tougher. It depends on our universal set U and the set A.

Clearly, $|A^C| \leq |U|$ since $A^C \subseteq U$ is always true (every set is a subset of the universal set). But it's hard to figure anything else out.

Cartesian Product

In finite sets, we have that $|A \times B| = |A| \cdot |B|$. Think about why.

But with countable sets, how do we multiply countable cardinality?

The set of particular interest is: $\mathbb{N} \times \mathbb{N}$. What is $|\mathbb{N} \times \mathbb{N}|$?

A Visual

The following grid describes $\mathbb{N} \times \mathbb{N}$ really well.

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)	$\cdots$
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)	$\cdots$
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)	$\cdots$
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)	$\cdots$
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)	$\cdots$
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)	$\cdots$
(1,7)	(2,7)	(3,7)	(4,7)	(5,7)	(6,7)	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Maybe you can see the issue. It goes to infinity in all directions. Can we come up with an algorithm to count up every pair?

A Hint

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)	$\cdots$
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)	$\cdots$
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)	$\cdots$
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)	$\cdots$
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)	$\cdots$
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)	$\cdots$
(1,7)	(2,7)	(3,7)	(4,7)	(5,7)	(6,7)	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Remember how we had to handle the integers? We need to reach every number after enough time. We can't choose one direction and 'turn around' at the end.

Also, remember the tip about putting every number in one long line and the dots only going in one direction.

The Answer

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)	$\cdots$
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)	$\cdots$
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)	$\cdots$
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)	$\cdots$
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)	$\cdots$
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)	$\cdots$
(1,7)	(2,7)	(3,7)	(4,7)	(5,7)	(6,7)	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

When I learned this technique, we called it diagonalization (this is actually incorrect but for some reason everyone understood it?), but there's a better name to describe this. Dovetailing!

Okay, time to pull out the animation budget.

Dovetailing

Dovetailing is the technique of arranging an infinite multidimensional grid in a sequence by counting/iterating each 'diagonal.'

There are a few things to note:

1: Diagonal is loosely defined. For intuition, there's no issue but for proofs, we need to rigourously define this.

2: Yes, this works in higher dimensions. I'll leave that as a worksheet question.

Proving $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

Proof. We start by arranging every element in $\mathbb{N} \times \mathbb{N}$ in a sequence as follows: First, group arbitrary elements of form (a, b) by $a + b$. So, $(3, 2)$ is grouped with $(4, 1)$ and all these will be in group five. Denote this as G_5.

Second, we iterate through each group starting at group two or G_2 since $1 + 1 = 2$. Note, each group has a finite number of elements since $|G_n| \leq n = a + b$ for valid a, b. Thus, we will reach every group after enough time. □

Proving $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

Proof. Continued.

Our sequence is $G_2, G_3, G_4, \ldots =$

$$\underbrace{(1,1)}_{G_2}, \underbrace{(1,2),(2,1)}_{G_3}, \underbrace{(3,1),(2,2),(1,3)}_{G_4}, (1,4), \ldots.$$

Now we'll show that $\mathbb{N}$ maps to this sequence bijectively.

Regarding surjectivity, if $(a,b) \in \mathbb{N} \times \mathbb{N}$, then $(a,b) \in G_{a+b}$. Thus, it gets mapped to.

Regarding injectivity, since every element can only be in one group and we don't allow duplicate elements in groups, we must have that each element is only mapped to once.

Thus, this mapping/function is bijective. □

A Quick Note

Here, we didn't explicitly define f. I know this is hypocritical, but in this case, it's hard to come up with f in a smart way. It's incredibly diffuclt to devise a rule for f. Thus, we argued it by other means.

It all comes down to rigor. If we can define f, we should! But in cases when we can't define f, we have no choice but to argue it like the previous proof.

Limits Of Dovetailing

There are a few limits to dovetailing.

Firstly, this won't work for infinite dimensions, only finite dimensions. So we can't use dovetailing to find $|\mathbb{N} \times \mathbb{N} \times \cdots|$.

Secondly, we need to figure out how to arrange our elements in a grid or by diagonals/groups.

Why Learn Dovetailing?

This technique doesn't come up often in mathematics. So why learn it?

I like to believe: Because dovetailing is cool. It's one of the most fun and thought provoking things in the first half.

But in actuality, it probably has to do with 'thinking outside the box.' It's problem solving at its finest. Knowing how to use techniques like these are what makes you better at math and problem solving.

Even if you never use this tool, knowing how it works can never hurt you.

Okay, One More!

What's $|\mathbb{Q}|$?

And $\mathbb{Q}$ is special. Between any two elements, we can find another element. For example, between $\frac{21}{5}$ and $\frac{22}{5}$, we can find something in between. For example, $\frac{43}{10}$. We say that $\mathbb{Q}$ is dense.

So, we need to come up with a smart way to list every element in $\mathbb{Q}$. How would we do so?

Extending $\mathbb{Z}$

Here's one way we can think of $\mathbb{Q}$.

$\cdots$	$\frac{-2}{1}$	$\frac{-1}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\cdots$
$\cdots$	$\frac{-2}{2}$	$\frac{-1}{2}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\cdots$
$\cdots$	$\frac{-2}{3}$	$\frac{-1}{3}$	$\frac{0}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\cdots$
$\cdots$	$\frac{-2}{4}$	$\frac{-1}{4}$	$\frac{0}{4}$	$\frac{1}{4}$	$\frac{2}{4}$	$\cdots$
$\cdots$	$\frac{-2}{5}$	$\frac{-1}{5}$	$\frac{0}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\cdots$
$\cdots$	$\frac{-2}{6}$	$\frac{-1}{6}$	$\frac{0}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\cdots$
$\cdots$	$\frac{-2}{7}$	$\frac{-1}{7}$	$\frac{0}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$\cdots$
$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Removing Duplicates

But remember, $\mathbb{Q}$ has to be in lowest form. So, we'll remove duplicates.

$\cdots$	$\frac{-2}{1}$	$\frac{-1}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\cdots$
$\cdots$		$\frac{-1}{2}$		$\frac{1}{2}$		$\cdots$
$\cdots$	$\frac{-2}{3}$	$\frac{-1}{3}$		$\frac{1}{3}$	$\frac{2}{3}$	$\cdots$
$\cdots$		$\frac{-1}{4}$		$\frac{1}{4}$		$\cdots$
$\cdots$	$\frac{-2}{5}$	$\frac{-1}{5}$		$\frac{1}{5}$	$\frac{2}{5}$	$\cdots$
$\cdots$		$\frac{-1}{6}$		$\frac{1}{6}$		$\cdots$
$\cdots$	$\frac{-2}{7}$	$\frac{-1}{7}$		$\frac{1}{7}$	$\frac{2}{7}$	$\cdots$
$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

If we go further right or left, we'd have to remove elements like $\frac{3}{6}$ or $-\frac{3}{3}$.

For Consideration

In the above picture, the grid extends out in three directions. Left, right and down. Can we turn this into a grid that only extends out in two directions?

We can. Act like the negatives don't exist, and every time we count a positive value, add in the negative one for free.

A Second Way To Count $\mathbb{Q}$

We use 'total' value. For example, $\frac{4}{3}$ has a total value of 7. $\frac{1}{2}$ has a total value of 3.

Yeah, all we do is add the numerator and denominator.

And this seems awfully similar... Let's see how this proof ends up being different than the one for $|\mathbb{N}| \times |\mathbb{N}|$

Proving $|\mathbb{Q}| = |\mathbb{N}|$

Proof. We'll arrange $\mathbb{Q}$ into a sequence as follows. First, we put every element in a group dependent on the sum of the absolute value of their numerator and denominator.

For example, $\frac{a}{b}$ will get put into $G_{|a|+|b|}$. The reason we need to consider the absolute value is because otherwise we have negative groups, which gets annoying to deal with.

Then, we'll count/iterate through every group as follows: $G_1, G_2, G_3, G_4, \ldots$. $\square$

Proving $|\mathbb{Q}| = |\mathbb{N}|$

Proof. Continued.
However, we still encounter a problem of duplicates. When creating G_n, don't include elements that aren't in lowest forms.

For example, $\frac{2}{2} \notin G_4$. This will ensure injectivity.

Finally, we note that each G_n is finite as $|G_n| < 2n$. Thus, every group is reached by our algorithm. □

Proving $|\mathbb{Q}| = |\mathbb{N}|$

Proof. Continued.

Our sequence is $G_1, G_2, G_3, G_4, \ldots =$

$$\underbrace{\frac{0}{1},}_{G_1} \underbrace{\frac{1}{1}, -\frac{1}{1},}_{G_2} \underbrace{\frac{1}{2}, \frac{2}{1}, -\frac{1}{2}, -\frac{2}{1},}_{G_3} \underbrace{\frac{3}{1}, \frac{1}{3}, -\frac{3}{1}, -\frac{1}{3},}_{G_4} \frac{1}{4}, \ldots$$

Now we'll show that $\mathbb{N}$ maps to this sequence bijectively.

Regarding surjectivity, if $\frac{a}{b} \in \mathbb{Q}$, then $\frac{a}{b} \in G_{|a|+|b|}$. Thus, it gets mapped to.

Regarding injectivity, since every element can only be in one group and we don't allow duplicate elements in groups, we must have that each element is only mapped to once.

Thus, this mapping/function is bijective. □

There We Have It

We have shown that $\mathbb{Q}$ has countable cardinality. Additionally, you can show that finite cartesian products of countable sets is still countable.

There's only two ways to proceed from here.

1: Countable cartesian products. We won't deal with this.

2: Cardinality of $\mathbb{R}$. This is what we'll do next time.