An Introduction to Mathematical Proofs

Pro And Contra

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When? Whenever You Watch

Truth Tables Return!

We use $x \Leftrightarrow y$ to indicate double-sided implications. And as seen in last video, we prove both $x \Rightarrow y$ and $y \Rightarrow x$, so it's the same as proving both of them.

x	y	$x \Rightarrow y$	$y \Rightarrow x$	$x \Rightarrow y \wedge y \Rightarrow x$	$x \Leftrightarrow y$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Understanding Implications

We deal with implications and double-sided implications a lot in Mathematics. So, we need to understand all the ways we can prove implications and math in general.

We'll introduce a technique we alluded to last time: the contrapositive

Contrapositive

Consider the statement: If it is raining, then kids stay inside.

Now, what do we know if the kids stay inside? We can't determine if it's raining or not. Recall the truth table. But what if kids are outside? Then it can't be raining!

If we want to prove $x \Rightarrow y$, this is equivalent to showing that $\neg y \Rightarrow \neg x$. This should remind you of double-sided implications!

Please, No More Truth Tables

Let x, y be mathematical statements. We'll list out $x \Rightarrow y, \neg x, \neg y$.

x	y	$x \Rightarrow y$	$\neg x$	$\neg y$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

I Need More Original Title Names

Now, we'll list out $\neg y \Rightarrow \neg x$

x	y	$x \Rightarrow y$	$\neg x$	$\neg y$	$\neg y \Rightarrow \neg x$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Logical Equivalence

Now, we'll list out $\neg y \Rightarrow \neg x$

x	y	$x \Rightarrow y$	$\neg x$	$\neg y$	$\neg y \Rightarrow \neg x$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

$x^2 + 1$ prime $\Rightarrow x = 2k, \exists k \in \mathbb{Z} \vee x = 1$

You can solve this proof either directly or by contrapositive. I'll do contrapositive to showcase this. For this proof, try to read the proof and understand why everything makes sense. Pause the video!

Proof. We'll show the contrapositive: If x is odd and $x > 1$, then $x^2 + 1$ is composite. Let $x = 2l + 1$ for some $l \in \mathbb{Z}$. Then we have:

$$
\begin{aligned}
x^2 + 1 &= (2l+1)^2 + 1 \\
&= 4l^2 + 4l + 1 + 1 \\
&= 2(2l^2 + 2l + 1)
\end{aligned}
$$

Thus, $x^2 + 1$ is even, and since $x > 1, x^2 + 1 > 2$ so $x^2 + 1$ is composite. $\square$

Essentially,

$x \Rightarrow y$ is the same as $\neg y \Rightarrow \neg x$.

They're equivalent, like how definitions are equivalent, because they have the same truth table.

So you can prove any implication either directly or by contrapositive.

Contradiction

If you've followed with the worksheets, you've seen this before.

We have an established set of rules. Then, we want to prove a proposition.

We can either prove the proposition, or show it can't be true.

How would we show a proposition can't be true? Because if it were true, it would break our established rules.

Prove That There Are Infinitely Many Primes

How would you do this? Showing there are infinitely many seems to be impossible to prove. So, let's try a proof by contradiction, or, assume that there aren't infinitely many primes and break a rule.

Proof. Assume for the sake of contradiction that there are finitely many primes. Then, we can list out every prime.

Let p_1 be the first prime, p_2 be the second prime and so on until p_n is the final prime.

So where's our contradiction? Remember, we want to show that there's infinitely many primes, so let's try to find a new prime number. □

Prove That There Are Infinitely Many Primes

Proof. Continued.

Consider $N = p_1 \cdot p_2 \cdots p_n$. Then, each prime divides N.

Now, what divides $N + 1$? If any prime were to divide $N + 1$, then it can't divide N. Understand why this works. Adding 1 to a number divisible by a prime p makes it so that p no longer divides it.

Since no primes divide $N + 1$, it must be a new prime number by definition.

Either we keep repeating this process to generate infinite primes, or recognize that our assumption of finitely many primes was false since the list was incomplete. Both lead to our desired result. $\square$

The Double Edged Sword

Contradiction is a very powerful tool. But like any tool, if used wrong, it can prove wrong results or invalidate a proof. For example, proving the wrong thing.

And usually, wrong proofs have a hard to catch step that ends up breaking the laws of mathematics. Thus, we arrive at a contradiction through broken jumps, not sound steps.

The Quintessential 'Wrong' Proof

Let $a = b$. Then we have:

$$\begin{aligned} ab &= b^2 \\ ab - a^2 &= b^2 - a^2 \\ a(b-a) &= (b-a)(b+a) \\ a &= b + a \\ 0 &= b \end{aligned}$$

Thus, everything is equal to 0? Math is fake? We didn't even assume a contradiction and broke the laws of mathematics!

The Issue

$$
\begin{aligned}
ab &= b^2 \\
ab - a^2 &= b^2 - a^2 \\
a(b-a) &= (b-a)(b+a) \\
a &= b + a \\
0 &= b
\end{aligned}
$$

Here, $b - a = 0$. Thus, we divide both sides by zero. Illegal! Clearly, both sides are equal to zero before the division but after the division, the error starts.