An Introduction to Mathematical Proofs
Conjunctions And Tables

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When? Whenever You Watch

A Quick Grammar Lesson

A conjunction is a word that connects two clauses.

Example I ate a cookie *and* drank some milk.

Example You can choose the red pill *or* the blue pill.

'and', 'or', 'whereas', 'but', 'after' are conjunctions. For our purposes, we only need 'and' and 'or.'

Clauses? No, Statements!

We can chain mathematical statements together with these two conjunctions, and get a different mathematical statement.

Example All primes are either odd or the number 2.

Example All integers are even and greater than zero.

All primes are either odd or the number 2.

Proof. Let p be a prime not equal to 2. Since p is prime, it isn't divisble by any other numbers other than p and 1.

So p isn't divisble by 2. Since $p \neq 2$, we must have that p can't be even, so it must be odd. $\square$

Another proof:

Proof. Let p be an even prime. Then, $p = 2k$ for some k.

If $k > 1$, then, we can divide p by 2 and it wouldn't be a prime. Thus, k must be 1 (If k were negative, p would be negative. If $k = 0$, then $p = 0$ isn't prime.), so $p = 2$. $\square$

Why Are The Above Proofs Correct?

We only need to show a single condition holds. So we can assume the other condition is false.

So, if condition two is false but then condition one is true, then the whole statement is true. The word 'or' does a lot of heavy-lifting.

Remember, we need to prove 'condition one or condition two', so we can leverage the 'or.'

Symbols

$\wedge$ means 'and' whereas $\vee$ means 'or.' These look awfully similar to the set operations so that might help you remember which means what.

The nice thing about these symbols is that we can use them in set builder as well, since the right hand side of the set builder acts like a mathematical statement.

Consider the set $A = \{ x : x$ is an integer, x is greater than 3 and x is even$\}$. How can we turn the right hand side into just symbols?

Symbols Continued

Consider the set $A =\{\ x\ :\ x$ is an integer, x is greater than 3 and x is even$\}$. How can we turn the right hand side into just symbols?

It's as follows:
$A = \{x : x \in \mathbb{Z} \wedge x \geq 3 \wedge \exists k \in \mathbb{Z}, x = 2k\}$

Unlike in English where we use commas with more than 3 items, we just spam more 'ands.'

To Or Not To

$\neg$: The negation symbol. So, if we want x is not even, we'd write it as $\neg \exists k \in \mathbb{Z}, x = 2k$, or, there does not exist k so that x is even.

But what's interesting is that there's also another way to represent this mathematical statement: $\forall k \in \mathbb{Z}, x \neq 2k$.

So there's many ways to write out a mathematical statement even in symbols!

Negate This!

Negate this statement: $\exists x \in \mathbb{Z}$ such that x is prime and $x \geq 4$.

One at a time here: The opposite of exists is for all. The opposite of x is prime is x is composite. The opposite of $x \geq 4$ is $x < 4$. Okay, but we're missing something. What's the opposite of 'and'?

$x = 5$ satisfies the statement above. Why? Because it's both prime and greater than 4. So what if we want to find a number so that the statement is true? We can either make it composiite or make it less than 4. So, we flip the 'and' into an 'or'

DeMorgan's Law

Negate this statement: $\exists x \in \mathbb{Z}$ such that x is prime and $x \geq 4$.

We get: $\forall x \in Z, x$ is composiite *or* $x < 4$.

If X and Y are mathematical statements, then we have:

$$\neg(X \wedge Y) = \neg X \vee \neg Y$$

This is called DeMorgan's Law, and it's pretty important.

Okay, This Is Still Abstract

We'll use what we call truth tables to explain how all these work with each other.

Humble Beginnings

Let x be a mathematical statement. We don't know whether it's true or false, so we'll write out all the possibilites of x in a table.

x
T
F

Let's Get A Friend

Let y be another mathematical statement. We don't know whether it's true or false, so we'll write out all the possibilites of y in the table. But now, we have 4 rows instead of 2, because we have 4 combinations.

x	y
T	T
T	F
F	T
F	F

Let's See How Close They Are

Now, we can consider other mathematical statements. For example, we could find the values of x and y, x or y

x	y	$x \wedge y$	$x \vee y$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Let's Get Some Opposites In Here

We know that $\neg x$ would be the opposite of x. So, if x is true, then $\neg x$ is false and vice versa. Let's add those in here.

x	$\neg x$	y	$\neg y$	$x \wedge y$	$x \vee y$
T	F	T	F	T	T
T	F	F	T	F	T
F	T	T	F	F	T
F	T	F	T	F	F

Now, We Flip Ands and Ors

Let's add $\neg(x \wedge y)$ and $\neg(x \vee y)$ and see what happens!

x	$\neg x$	y	$\neg y$	$x \wedge y$	$x \vee y$
T	F	T	F	T	T
T	F	F	T	F	T
F	T	T	F	F	T
F	T	F	T	F	F

$\neg(x \wedge y)$	$\neg(x \vee y)$
F	F
T	F
T	F
T	T

It's DeMorgan Time

Let's add $\neg x \wedge \neg y$ and $\neg x \vee \neg y$ and see what happens!

x	y	$x \wedge y$	$x \vee y$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

$\neg x$	$\neg y$	$\neg(x \wedge y)$	$\neg(x \vee y)$	$\neg x \wedge \neg y$	$\neg x \vee \neg y$
F	F	F	F	F	F
F	T	T	F	F	T
T	F	T	F	F	T
T	T	T	T	T	T

Do You See It?

$\neg x$	$\neg y$	$\neg(x \wedge y)$	$\neg(x \vee y)$	$\neg x \wedge \neg y$	$\neg x \vee \neg y$
F	F	F	F	F	F
F	T	T	F	F	T
T	F	T	F	F	T
T	T	T	T	T	T

Do You See It?

$\neg x$	$\neg y$	$\neg(x \wedge y)$	$\neg(x \vee y)$	$\neg x \wedge \neg y$	$\neg x \vee \neg y$
F	F	F	F	F	F
F	T	T	F	F	T
T	F	T	F	F	T
T	T	T	T	T	T