An Introduction to Mathematical Proofs

Composition

Who? Fahad Hossaini

When? Whenever You Watch

We Love Functions!

We'll continue to talk about functions.

In particular, we'll be looking at function composition, or, combining functions. Then, we'll see how function composition and (in/sur/bi)jectivity are related.

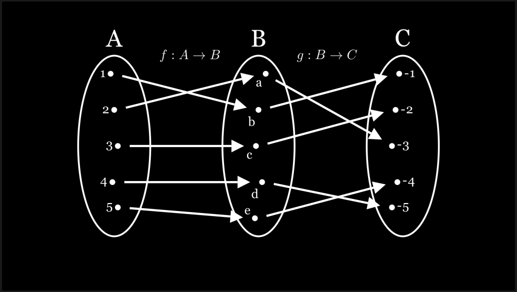
First, A Visual

Two Things To Note

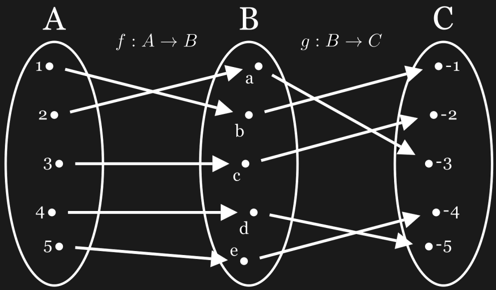
1: In this case, $A \neq B \neq C$. This is to make sure there's no confusion among sets. But later on, you might have $A = B = C$ and in those cases, make sure not to get tripped up.

2: Yes, both functions are bijective (for now). We'll up the difficulty soon. :)

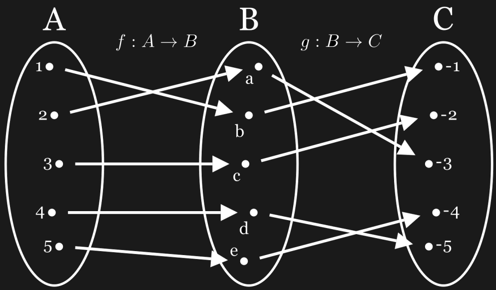
Direct Screenshot



Let's Pretty It Up



First, Order

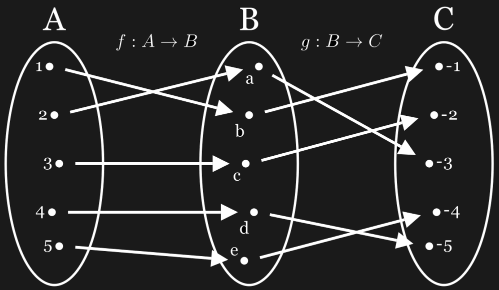


We want a function $h : A \to C$. Do we first apply f and then g or g and then f?

Our inputs are from A, so we first apply f and then g.

So, $h : A \to C, h(x) = g(f(x)) = (g \circ f)(x)$. Or, $h = g \circ f$.

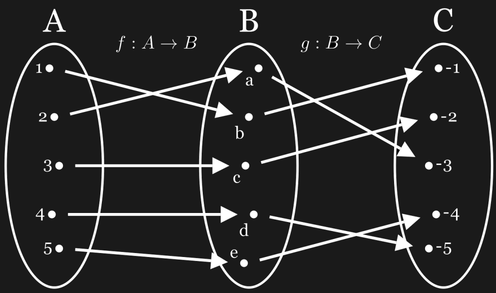
Be Careful!



Firstly, $h = g \circ f \neq f \circ g$. These are two completely different things. In fact, $f \circ g$ doesn't make sense in this context!

Secondly, we apply/read from right to left. $g \circ f$ means we apply f first and then g.

Now, Some Examples

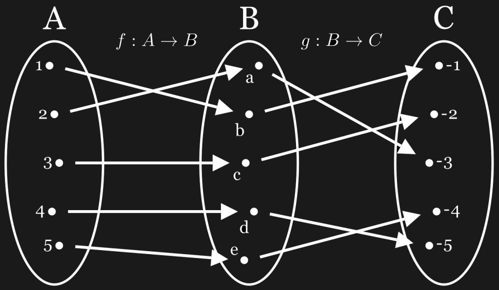


Example What's $h(1)$?

Example What's $h(2)$?

Example What's $h(4)$?

Now, Some Answers



Example $h(1) = g(f(1)) = g(b) = -1$

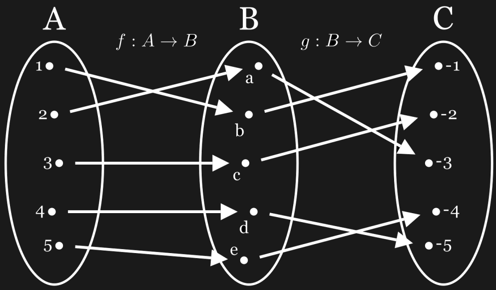
Example $h(2) = g(f(2)) = g(a) = -3$

Example $h(4) = g(f(4)) = g(d) = -5$

What's Next?

Now that we're somewhat comfortable with function composition, we'll look at bijections!

Investigating h^{-1}



We can see that f and g are both bijections. Thus, f^{-1} and g^{-1} exist. But does h^{-1} exist?

Intuitively, if we can uniquely map an element from A to C via h, then h is a bijection and should be invertible!

Property 1

If $f : A \to B$ and $g : B \to C$ are both bijective, then $g \circ f$ is also bijective!

We can prove this!

Proof. For sake of notation, let $h = g \circ f$. □

Injectivity

Proof. Continued.

We'll start by showing injectivity.

If $x_1 \neq x_2$, then $h(x_1) \neq h(x_2)$. We have:

$$
\begin{aligned}
x_1 &\neq x_2 & \\
f(x_1) &\neq f(x_2) & \text{Since f is injective} \\
g(f(x_1)) &\neq g(f(x_2)) & \text{Since g is injective} \\
h(x_1) &\neq h(x_2) &
\end{aligned}
$$

Note that since we only require f and g to be injective, we also proved the following:

If $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f$ is also injective! □

Surjectivity

Proof. Continued.
We'll continue by showing surjectivity.

Let $z \in C$. Then, let $x = f^{-1}(g^{-1}(z))$. Note that both inverses exist since f and g are bijective. We have:

$$
\begin{aligned}
h(x) &= g(f(f^{-1}(g^{-1}(z)))) \\
&= g(g^{-1}(z)) && \text{Since } f \circ f^{-1} \text{ cancel} \\
&= z && \text{Since } g \circ g^{-1} \text{ cancel}
\end{aligned}
$$

Thus, h is both injective and surjective, so h must be bijective.

□

A Formal Note

If we have an injective function, we can restrict the codomain to make the range and codomain line up. Thus, we can get a bijective function by changing the codomain.

Likewise, if we have a surjective function, we can restrict the domain to ensure Injectivity and end with a bijective function by changing the domain.

In the case that f and g are only surjective, restrict the domain of both f and g so that f and g become bijective. Since the restricted domain is a subset of our original domain, we can produce a valid $x \in A$ so that $h(x) = z$. So, we also have that:

If $f : A \to B$ and $g : B \to C$ are both surjective, then $g \circ f$ is also surjective!

We've Proved The Following:

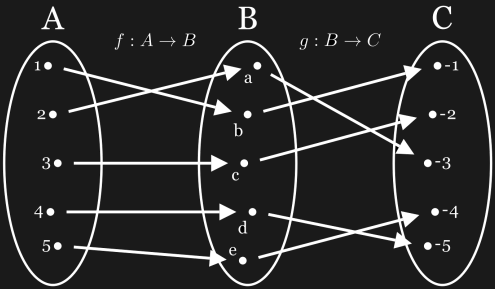
If $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f$ is also injective!

If $f : A \to B$ and $g : B \to C$ are both surjective, then $g \circ f$ is also surjective!

If $f : A \to B$ and $g : B \to C$ are both bijective, then $g \circ f$ is also bijective!

We can continue investigating these relationships further or figure out how h^{-1} behaves. We'll do the latter.

Can You Spot h^{-1}?



Since h is bijective, it's invertible. Thus, we have that: $h^{-1} : C \to A$. But what's the formula?

As we showed in our surjectivity proof, $h^{-1} = f^{-1} \circ g^{-1}$.

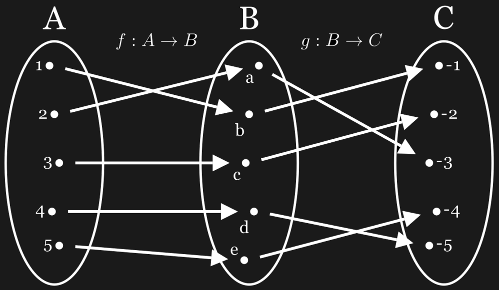
Yes, while $h = g \circ f$, think about how the order goes for h^{-1}.

Think Of Domain and Codomain

$g^{-1}: C \to B$. Since h^{-1} takes elements from C, it must be the case that g^{-1} is applied first.

The rule of thumb is this: if $h = g \circ f$, then $h^{-1} = f^{-1} \circ g^{-1}$.

More Examples

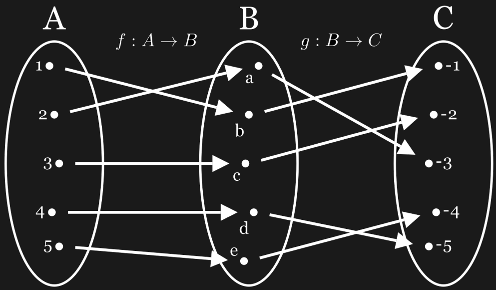


Example What's $h^{-1}(-1)$?

Example What's $h^{-1}(-3)$?

Example What's $h^{-1}(-4)$?

More Answers



Example $h^{-1}(-1) = f^{-1}(g^{-1}(-1)) = f^{-1}(b) = 1$

Example $h^{-1}(-3) = f^{-1}(g^{-1}(-3)) = f^{-1}(a) = 2$

Example $h^{-1}(-4) = f^{-1}(g^{-1}(-4)) = f^{-1}(e) = 5$

Minimal Conditions

Lastly, we'll look at an interesting problem.

If $h = g \circ f$ is a bijection, what must f, g be? Or, what are the least restrictive conditions on f and g so that h could be a bijection?

This is different then what we solved before. We showed before that if f and g are bijections, so is their composition. You can think of that as a 'for all' statement. Whereas this new question is more like a 'there exists' statement.

Start With g

We know that $h : A \to C$. Since $g : B \to C$, in order for h to reach all of C, g must be surjective.

To see this, assume that g wasn't surjective. Then, $\text{range}(h) \subseteq \text{range}(g) \neq \text{codomain}(g)$. Thus, h would never be bijective, a contradiction.

But then what about injectivity?

I can't come up with better titles

Let's say g is not injective. Then, we can find some pair $x_1, x_2 \in B$ so that $f(x_1) = f(x_2)$. While this might seem like an issue, I'll show you that it's not!

Did You See That?

All we did was add a point in f and in this case, h was still bijective!

But before we get too ahead, let's analyze f and its minimal conditions.

Continue With f

I claim that f must be injective. For the sake of contradiction, assume f is not injective. Then there exists $x_1 \neq x_2$ so that $f(x_1) = f(x_2)$. Let $y = f(x_1)$.

Then, since h is bijective, we have that $x_1 \neq x_2 \Rightarrow h(x_1) \neq h(x_2)$:

$$h(x_1) \neq h(x_2)$$
$$g(f(x_1)) \neq g(f(x_2))$$
$$g(y) \neq g(y)$$

Contradiction! So f must be injective. But f doesn't need to be surjective as seen before.

One Last Note

I'll alter the function we've been working with so you can see something important.

Keep This In Mind

We proved that if $g \circ f$ is bijective, g must be surjective and f must be injective.

This is NOT the same as: If g is surjective and f is injective, then $g \circ f$ is bijective. This is FALSE.

What we deduced earlier is that if both f, g are bijections, then their composition is a bijection as long as the domain and codomain line up.

One of these forces condition on h while the other forces a condition on f and g.