Answers are found at the bottom of the worksheet on a seperate page.

**Question 1:** Show that $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is countable.

**Question 2:** Find functions that verify the following facts:

1. $|\{1, 2, 3\}| = |\{1, 4, 9\}|$,
2. $|\mathbb{N}| \leq |\mathbb{Z}|$,
3. $|[-2020, 2020]| \leq |(-2, 2)|$.

**Question 3:** Find a bijective function from $[0, 1]$ to $[0, 1)$. This is actually really hard.

Hint: We need to find a way to send 1 into $[0, 1)$ in a bijective manner. The best way to do this is by creating a piecewise function that keeps most values of $x \in [0, 1]$ unchanged but somehow affects 1 and a few other (countably many) values. This is still hard even with this hint.

**Question 4:** We'll introduce the notion of a power set.

A power set of a set $A$ is all possible subsets of $A$. We denote it as follows:

$\mathcal{P}(\{1, 2, 3\}) = \{\{1\}.\{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \varnothing\}$

Show that $|A| \leq |\mathcal{A}|$ by finding an injective function from $A$ to $\mathcal{P}(A)$ or a surjective function from $\mathcal{P}(A)$ to $A$.

**Question 5:** Let $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N} \cup \{\text{DNE}\}$, be defined by $f(A)$ is the minimum/smallest element of $A$, if $A \neq \emptyset$ and $f(\emptyset) = \text{DNE}$.

Let $g : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{N})$ be the (restriction) function $g(A) = A \cap \mathbb{N}$.

1. Compute $f(\{2, 3, 5\})$.
2. Compute $f(\mathbb{N})$.
3. Compute $f(\emptyset)$.
4. Compute $f \circ g([1.5, 4) \cup [4.5, 5.5])$.
5. Compute $f \circ g(\mathbb{R})$.
6. Compute $f \circ g([-2, 0])$.
7. In words, explain what the function $f \circ g$ does.
8. Prove that if $A \subseteq B \subseteq \mathbb{R}$ and $A \cap \mathbb{N} \neq \emptyset$, then $f \circ g(A) \geq f \circ g(B)$.

**Question 6:** Construct a bijection that shows that the following sets are countable.

1. $\{-2000, -1999, \ldots, 1, 0\} \cup \mathbb{N}$.
2. $\mathbb{Z} \cup \{x + \frac{1}{2} : x \in \mathbb{Z}\}$.
3. $\{1, 10, 100, 1000, \ldots, \}$
4. $\mathbb{N} \times \mathbb{Q}$.

**Question 7:**

Identify which of the following sets are finite, countable, or uncountable. Give a short explanation for your choice (a complete proof is not necessary).

1. $\{1, 2, 3, \ldots, 2019\}$.

2. $\mathbb{Z}$.
3. $A \cup \mathbb{R}$, where $A$ is any set.
4. $\mathbb{Q} \cap [0, 1]$.

**Remark:** It's possible to show that $|A| < |\mathcal{P}(A)|$. When I learned this, we assumed for the sake of contradiction that a surjective function exists from $A$ to $[P](A)$.

Then, you can derive a contradiction where an element gets mapped to if and only if it isn't mapped to (if I remember correctly).

What really matters is the following consequence: $|\mathbb{R}| < |\mathcal{P}(\mathbb{R})|$. So then what's the cardinality of $\mathcal{P}(\mathbb{R})$? We just created a third infinty that's different in size and *larger* than uncountable.

But what if we apply the power set operation to the power set? So, $|\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))|$...

So we can create *countably many* different sizes of infinity, each larger than the one before...

And just this time, a second remark. Does there exists a cardinality that's between countable and uncountable? Well, we use the ZFC set axioms, which is the basis of all of set theory. So, we should be able to derive whether such a cardinality exists from these set of axioms. I mean, these are the most commonly accepted set of axioms for set theory! So does such a cardinality exist?

Well, it's an underivable fact. As in, this fact is not dependent on the axioms. As in, you can decide if such a cardinality exists... This is called the continuum hypothesis.

**Answer 1:**

*Proof.* This is very similar to the $\mathbb{N} \times \mathbb{N}$ case so I will only include necessary details.

Send elements of form $(a, b, c)$ into $G_{a+b+c}$.

Expand $G_3, G_4, G_5, ldots$ and map to this sequence.

A mapping to this sequence will be surjective because every element belongs in a group. The same mapping will be injective because we contain no duplicates.

□

**Answer 2:**

1. $|\{1, 2, 3\}| = |\{1, 4, 9\}|$. Let $f(1) = 1, f(2) = 4, f(3) = 9$ or $f(x) = x^2$. This is bijective.
2. $|\mathbb{N}| \leq |\mathbb{Z}|$. Let $f(x) = x$. This is injective.
3. $|[-2020, 2020]| \leq |(-2, 2)|$. Let $f(x) = \frac{x}{2020}$. This is injective.

**Answer 3:**

*Proof.* The function $f : [0, 1] \to [0, 1)$ given by: $\begin{cases} \frac{x}{2} & \text{if } x \in \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\} \\ x & \text{otherwise} \end{cases}$ is a bijective function. □

**Answer 4:**

*Proof.* We'll construct a injective function from $A$ to $\mathcal{P}(A)$.

Recall that the codomain is a set of sets. Thus, our functions maps elements to sets. Let $f : A \to \mathcal{P}(A)$.

Then, $f(x) = \{x\}$ is an injective function.

To see this, let $x_1 \neq x_2 \in A$. Then, we see that $f(x_1) = \{x_1\} \neq \{x_2\} = f(x_2)$.

Thus, $|A| \leq |\mathcal{P}(A)|$.

□

**Answer 5:**

1. $f(\{2, 3, 5\}) = 2$ as 2 is the smallest element in $\{2, 3, 5\}$
2. $f(\mathbb{N}) = 1$
3. $f(\varnothing) = DNE$
4. $f \circ g([1.5, 4) \cup [4.5, 5.5]) = f(\{2, 3, 4, 5\}) = 2$
5. $f \circ g(\mathbb{R}) = f(\mathbb{N}) = 1$
6. $f \circ g([-2, 0]) = f(\varnothing) = DNE$.
7. In words, explain what the function $f \circ g$ does: If it exists, it picks out the smallest natural number from A.
8. Prove that if $A \subseteq B \subseteq \mathbb{R}$ and $A \cap \mathbb{N} \neq \emptyset$, then $f \circ g(A) \geq f \circ g(B)$. Note that both values of the composition are defined, and are numbers. No DNEs. If B is a superset, then the minimum element can only go down, it can't go up! Since the intuition is most important, no proof will be included.

**Answer 6:**

1. Define $f : \{-2000, -1999, \dots, 1, 0\} \cup \mathbb{N} \to \mathbb{N}$ by $f(x) = x + 2001$. You can prove this is bijective.
2. Define $f : \mathbb{Z} \cup \{x + \frac{1}{2} : x \in \mathbb{Z}\} \to \mathbb{Z}$ by $f(x) = 2x$. You can prove this is bijective, and since $\mathbb{Z}$ is countable, so is $\{x + \frac{1}{2} : x \in \mathbb{Z}\}$.
3. Define $f : \mathbb{N} \to \{1, 10, 100, 1000, \dots, \}$ by $f(n) = 10^{n-1}$. You can prove this is bijective.
4. Since we showed that $\mathbb{Q}$ is countable, we can dovetail the following grid using our enumeration/sequence of $\mathbb{Q}$ to show that $\mathbb{N} \times \mathbb{Q}$ is countable:

| | | | | | | |
|---|---|---|---|---|---|---|
| $(1, \frac{0}{1})$ | $(2, \frac{0}{1})$ | $(3, \frac{0}{1})$ | $(4, \frac{0}{1})$ | $(5, \frac{0}{1})$ | $(6, \frac{0}{1})$ | $\cdots$ |
| $(1, \frac{1}{1})$ | $(2, \frac{1}{1})$ | $(3, \frac{1}{1})$ | $(4, \frac{1}{1})$ | $(5, \frac{1}{1})$ | $(6, \frac{1}{1})$ | $\cdots$ |
| $(1, -\frac{1}{1})$ | $(2, -\frac{1}{1})$ | $(3, -\frac{1}{1})$ | $(4, -\frac{1}{1})$ | $(5, -\frac{1}{1})$ | $(6, -\frac{1}{1})$ | $\cdots$ |
| $(1, \frac{2}{1})$ | $(2, \frac{2}{1})$ | $(3, \frac{2}{1})$ | $(4, \frac{2}{1})$ | $(5, \frac{2}{1})$ | $(6, \frac{2}{1})$ | $\cdots$ |
| $(1, \frac{1}{2})$ | $(2, \frac{1}{2})$ | $(3, \frac{1}{2})$ | $(4, \frac{1}{2})$ | $(5, \frac{1}{2})$ | $(6, \frac{1}{2})$ | $\cdots$ |
| $(1, -\frac{2}{1})$ | $(2, -\frac{2}{1})$ | $(3, -\frac{2}{1})$ | $(4, -\frac{2}{1})$ | $(5, -\frac{2}{1})$ | $(6, -\frac{2}{1})$ | $\cdots$ |
| $(1, -\frac{1}{2})$ | $(2, -\frac{1}{2})$ | $(3, -\frac{1}{2})$ | $(4, -\frac{1}{2})$ | $(5, -\frac{1}{2})$ | $(6, -\frac{1}{2})$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

**Answer 7:**

1. $\{1, 2, 3, \dots, 2019\}$. Finite
2. $\mathbb{Z}$. Countable.
3. $A \cup \mathbb{R}$, where $A$ is any set. Uncountable, since it is a superset of R which is uncountable.
4. $\mathbb{Q} \cap [0, 1]$. Countable, since Q is countable, and intersection yields cardinality at most of the smallest set.