Answers are found at the bottom of the worksheet on a seperate page.

Question 1: Let $f : A \to B$. Which of the following two statements means that f is injective? What does the other one mean? Write the contrapositives of both implications.

1. $\forall x_1, x_2 \in A$, if $x_1 = x_2$, then $f(x_1) = f(x_2)$.
2. $\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Question 2: Give specific points a, b that show that these functions are not injective:

1. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \frac{1}{x^2}$.
2. $g : (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by $g(x) = \cos(x)$.
3. $h : \mathbb{N} \to \mathbb{R}$ given by $h(n) = (-1)^n$.

Question 3: Give specific points y the codomain of these functions that show that these functions are not surjective:

1. $f : \mathbb{R} \setminus \{0\} \to [0, \infty)$ given by $f(x) = \frac{1}{x^2}$.
2. $g : [\frac{-\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ given by $g(x) = \cos(x)$.
3. $h : \mathbb{N} \to [-1, 1]$ given by $h(n) = (-1)^n$.

Question 4: Show that there are 8 different functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Hint: Write your function in the following form: $(f(1), f(2), f(3))$. Then, counting these functions will be easier.

Question 5: Show that there are 6 different <u>surjective</u> functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Question 6: Show that there are <u>NO</u> <u>injective</u> functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Question 7: Challenge: Generalize the 3 previous exercises to functions with domain $\{1, 2, \ldots, 10\}$, and codomain $\{0, 1\}$.

Question 8: We'll be exploring an interesting concept in this question.

First, we recall that the set $\{1, 2, 3\}$ has $2^3 = 8$ subsets. Keep this in mind.

Second, we introduce something called the indicator function. Define $f : A \to \{0, 1\}$ by the following rule: $f(x) = 1$ if $x \in A$. Otherwise, $f(x) = 0$.

Now, let $A \subseteq \{1, 2, 3\}$. Do you see the relation between subsets and the previous questions via this indicator function?

As an example, let $A = \{1, 2\}$. Then, $f(1) = 1, f(2) = 1, f(3) = 0$. Concatenating these together yields the following binary string: 110. Or, if we think this as a tuple, we get: $(1, 1, 0)$.

Now, the question. Show that there's a bijection between subsets of $\{1, 2, 3\}$ and all possible functions that map $\{1, 2, 3\}$ to $\{0, 1\}$. You'll probably have to use the indicator function for this.

As an extra challenge, generalize this to subsets of a set of form: $\{1, 2, \ldots, k\}$ and all possible functions that map $\{1, 2, \ldots, k\}$ to $\{0, 1\}$.

Question 9: Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. For each condition, give an example of functions $f_i : A \to B$ and $g_i : B \to A$ with the following properties, or explain why it is impossible.

1. g_1 is the inverse of f_1.

2. $f_2 \circ g_2(x) = g_2 \circ f_2(x)$ for all x.
3. $f_3 \circ g_3(x) = x$ for all $x \in B$.
4. The range of $g_4 \circ f_4$ has 2 elements, but the range of $f_4 \circ g_4$ has 1.
5. The range of $g_5 \circ f_5$ has 3 elements, but the range of $f_5 \circ g_5$ has 2.

Answer 1:

1. is the definition of being a function. It is vertical line test (if you have seen this in any other classes). The contrapositive is:

$\forall x_1, x_2 \in A$, if $f(x_1) \neq f(x_2)$, then $x_1 \neq x_2$.

2. is the definition of injective. The contrapositive is:

$\forall x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Answer 2:

1. Example: $a = 1$ and $b = -1$.
2. Example: $a = \frac{-\pi}{4}$ and $b = \frac{\pi}{4}$.
3. Example: $a = 2$ and $b = 4$.

Answer 3:

1. Example: $y = 0$, because $\frac{1}{x^2} > 0$.
2. Example: $y = 2$, because $\cos x \leq 1 < 2$ for all $x \in \mathbb{R}$.
3. Example: $y = 0$, because h only has outputs -1 and 1.

Answer 4:

Proof. Write functions as $f(1)f(2)f(3)$ instead of $(f(1), f(2), f(3))$. So for example 110 is the function $f(1) = 1, f(2) = 1, f(3) = 0$. This notation suggests the (true) fact that the collection of all functions from $\{1, 2, \ldots, n\}$ to $0, 1$ is the collection of all binary strings of length n. This makes counting easier.

$$000, 001, 010, 011, 100, 101, 110, 111$$

□

Answer 5:

Proof. Everything above works EXCEPT 000 and 111. This reminds us of this important principle of counting: Instead of counting the good things, you can count the bad things.

□

Answer 6:

Proof. This is impossible because the domain has 3 elements, but the codomain has only 2 choices. One of the choices must be repeated (by the pigeonhole principle).

□

Once again, 'size' of sets play a huge role in determining what type of functions exist from one set to another.

Answer 7:

Proof. There are 2^n many functions, and all but 2 of them are surjective. Namely, the function that sends everything to 0 and the function that sends everything to 1. There are never any injective functions.

□

Answer 8:

We'll do the general case.

Proof. Let $\{1, 2, \ldots, k\}$ be given.

Define B to be the set of all binary strings of length k. By binary strings, we mean k many 0s and 1s. For example, if $k = 5$, then $B = \{00000, 00001, 00010, \ldots, 11110, 11111\}$.

Now, consider all subsets of $\{1, 2, \ldots, k\}$. We'll form a bijection from these subsets to B.

We'll use the indicator function to build our bijection. Let $A \subseteq \{1, 2, \ldots, k\}$. If $x \in A$, then $f(x) = 1$ and 0 otherwise.

Then, concatenate $f(1)f(2)\cdots f(k)$. Call this binary string b_A.

Now, $g : \mathcal{P}(\{1, 2, \cdots, k\}) : B$ where $\mathcal{P}(\{1, 2, \cdots, k\})$ is all subsets of $\{1, 2, \cdots, k\}$ defined by $g(A) = b_A$.

I claim this is a bijective function.

Regarding injective, by contrapositive, if $A_1 \neq A_2$, then some $x \in A_1$ is not in A_2 or vice versa. Thus, $f(x) = 1$ for A_1 but $f(x) = 0$ for some A_2 or vice versa. Thus, $b_{A_1} \neq b_{A_2}$. Thus, this function is injective.

Regarding surjective, let $b \in B$ be a binary string of length k. Then, let b_i be the i^{th} digit of b. If $b_i = 1$, then include i in A. Otherwise, let $i \notin A$. Then, by defintion, we have that $b_A = b$ as every digit is covered. Thus, the function is surjective and by extension, bijective.

Now, we'll show that the possible functions from $\{1, 2, \ldots, k\}$ to $\{0, 1\}$ is also bijective to B.

However, we can write functions as $f(1)f(2)\cdots f(k)$ and we showed before that this type of concatenation results in a bijective function.

Finally, since we can find a bijection from $\mathcal{P}\{1, 2, \ldots, k\}$ to B and from B to all possible functions between $\{1, 2, \ldots, k\}$ and $\{0, 1\}$, we must have that we can create a bijective function from $\mathcal{P}\{1, 2, \ldots, k\}$ to all possible functions between $\{1, 2, \ldots, k\}$ and $\{0, 1\}$.

□

Answer 9:

1. Possible (easily).
2. Impossible because they don't have the same domain!
3. Possible if g_3 is the inverse of f_3.
4. Possible: $f(1) = f(2) = 4, f(3) = 6$ and $g(4) = g(6) = 1, g(5) = 2$.
5. Impossible since the first condition forces both functions to be bijections. (This can be seen by contradiction.)